Lie Algebras Solutions

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Foreword

This solutions document is a companion to *Lie Algebras* by Fulton B. Gonzalez, as available at `www.tufts.edu/~fgonzale/`. These solution were typed throughout a semester long course using the version dated December 4th, 2007. Note that while there are 11 Chapters, only the first 8 of them contain exercises. This is intended for personal use, and I cannot guarantee the accuracy of every solution within.

The most recent version of these solutions can be found on my personal webite `www.asanchezmath.com`.

Chapter 1: Background Linear Algebra

1.1.2 $\Rightarrow$

Say $x \in U \cap W$. We must also have that $(-x)$ is in $U \cap W$ as well since vector spaces are closed under multiplication by constants. $(x) + (-x) = 0$.
Since $U + W$ is direct, must have $x = (-x) = 0$, so $U \cap W = \{0\}$.

$\Leftarrow$

Say for $u \in U$ and $w \in W$ we have $u + w = 0$. Then $u + w \in U \cap W$, then $u = (-w) \in W$, meaning $u \in U \cap W$, thus $u = 0$. $w = 0$ follows from there, and therefore $U + W$ is a direct sum.

1.1.4 Let $\{v_1, \dots, v_k\}$ be a basis of $U \cap W$.
Extend this to a basis $\{u_1, \dots, u_n, v_1, \dots, v_k\}$ of U.
Similarly, extend it to a basis $\{v_1, \dots, v_k, w_1, \dots, w_m\}$ of W.
$\{u_1, \dots, u_n, v_1, \dots, v_k, w_1, \dots, w_m\}$ is then a basis of $U + W$.
$\dim(U) = n + k$, $\dim(W) = k + m$,
$\dim(U \cap W) = k$, and $\dim(U + W) = n + k + m$

$$(n + k + m) = (n + k) + (k + m) - (k)$$

Note that if you have $U \bigoplus W$, then by Exercise 1.1.2, $k = 0$.

1.3.1 Let $B = \{v_1, \dots, v_n\}$, $B' = \{w_1, \dots, w_m\}$, and $B'' = \{u_1, \dots, u_k\}$

$T(v_j) = \sum_{h=1}^{m} a_{hj} w_h$, and $S(w_h) = \sum_{i=1}^{k} b_{ih} u_i$.

The (ih)-th entry of $M_{B''B}(S) = b_{ih}$ and the (hj)-th entry of $M_{B'B}(T) = a_{hj}$.

Via matrix multiplication, the (ij)-th entry of $M_{B''B}(S) M_{B'B}(T) = \sum_{h=1}^{m} b_{ih} a_{hj}$

On the other hand, by composing first, we get

$$S \circ T(v_j) = S\left(\sum_{h=1}^{m} a_{hj} w_h\right) = \sum_{h=1}^{m} a_{hj} S(w_h) = \sum_{h=1}^{m} \sum_{i=1}^{k} a_{hj} b_{ih} u_i = \sum_{i=1}^{k} \left(\sum_{h=1}^{m} b_{ih} a_{hj}\right) u_i$$

Thus the (ij)-th entry of $M_{B''B}(ST) = \sum_{h=1}^{m} b_{ih} a_{hj}$

Since they are equal in all entries, $M_{B''B}(ST) = M_{B''B}(S) M_{B'B}(T)$

1.3.4 1) $T : V \to W$ injective $\Leftrightarrow {}^tT : W^* \to V^*$ surjective.

$\Rightarrow$

Let $f : V \to \mathbb{F} \in V^*$. Take a basis $\{v_1, \dots v_n\}$. Since T is injective these map to n-linearly independent vectors, $\{T(v_1), \dots, T(v_n)\}$ in W.
Expand this to a basis of W, $\{T(v_1), \dots, T(v_n), w_1, \dots, w_k\}$
For all $w \in W$, $w = \sum_{i=1}^n a_i T(v_i) + \sum_{j=1}^k b_j w_j = T\left(\sum_{i=1}^n a_i v_i\right) + \sum_{j=1}^k b_j w_j$.

Let v be the unique vector in V given by $\sum_{i=1}^n a_i v_i$ and w' be the unique vector in $W/T(V)$ given by $\sum_{j=1}^k b_j w_j$, then that means $w = T(v) + w'$.

Define $g : W \to \mathbb{F}$ by $g(w) = L(T(v) + w') = f(v)$. Since decomposition into $T(v)$ and w are unique, this is a well defined linear functional.
Then ${}^tT(g)(v) = g \circ T(v) = g(T(v) + 0) = f(v)$, so ${}^tT(g) = f$,
Therefore, ${}^tT$ is surjective.

$\Leftarrow$

Say $\ker T \neq \{0\}$, then there exists $v \in \ker T$, $v \neq 0$ such that $T(v) = 0$.
from v we can get a basis $\{v, v_2, \dots, v_n\}$ of V.

Let $f(v) = 1$, and extend the function linearly to the basis to get $f \in V^*$.
Since ${}^tT$ is surjective, there exists a $g \in W^*$ such that $f = {}^tT(g)$

$$1 = f(v) = {}^tT(g)(v) = g \circ T(v) = g(0) = 0$$

This is a contradiction, so such a v cannot exist, and thus $\ker T = \{0\}$, and therefore T is injective.

2)
Define $(T(V))^\perp := \{f \in W^* : f \circ T(v) = 0, \forall v \in V\}$.
$f \in \ker {}^tT$ means that ${}^tT(f) = f \circ T = 0$. This means that for all $v \in V$, $f \circ T(v) = 0$, and thus we have that $\ker {}^tT = (T(V))^\perp$.

If $\{v_1, \dots, v_n\}$ are a basis of V, we can get a spanning set of $T(V)$, $\{T(v_1), \dots, T(v_n)\}$.
By throwing out a finite number $T(v_i)$'s, we can have this be a basis of T(V).
Expand this to a basis of W, $\{T(v_1), \dots, T(v_{n'}), w_1, \dots, w_k\}$ and take the dual basis $\{\alpha_1, \dots, \alpha_{n'}, \beta, \dots, \beta_k\}$ of W^*.
Notice the following: $\beta_j(T(v_i)) = 0$, so $\beta_i \in (T(V))^\perp$ Since the $T(v_i)$ span $T(V)$, the β_j must span $(T(V))^\perp$, and we have the following:

$\dim W^* = \dim T(V) + \dim(T(V))^\perp$

From rank-nullity, we have $\dim {}^tT(V^*) = \dim W^* - \dim \ker({}^tT)$.

$\dim {}^tT(V^*) = \dim W^* - (\dim W^* - \dim T(V)) = \dim T(V)$

3) An $m \times n$ matrix A can be thought of as the matrix of a linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ in the standard orthonormal basis.

The dimension of the column space of A is the same as the dimension of the image of the linear function T.

Row space can be thought of as the dimension of the column space of the tranpose of A. We know that $M_{B'^*,B^*}({}^tT) = {}^t(M_{B,B'}(T))$, so the dimension of the row space is actually the dimension of the image of ${}^tT$.

By part (b), we know that these two images have the same dimension, so the column space dimension must equal the row space dimension.

1.4.1 First we note that if we take n linearly dependent vectors, they do not have any n-dimensional volume and also have a zero determinant, so we can assume that we are taking $\{v_1, \dots, v_n\}$ linearly independent.

Let A be the matrix obtained by taking v_1 through v_n as columns of A.

We prove this by induction on the dimension. Say that the determinant of n vectors is the volume of the n parallelepiped whose sides are the column vectors. Let $\{v_1, \dots, v_{n+1}\}$ be the vectors along the side of an $(n+1)$-parallelepiped.

The volume of the $(n+1)$-parallelepiped is the volume of the base times the height. Here the base is the $(n-1)$ dimensional parallelepiped given by $\{v_1, \dots, v_n\}$ and the height is the projection of v_{n+1} onto the vector to n normal to the hyperplane spanned by $\{v_1, \dots, v_n\}$

First we note that adding a linear combination of v_i's to the v_{n+1} will not change the height by the following reason:
The height h is the length of the projection of v_{n+1} onto n, and the projection of any v_i onto n must be 0, since n is normal to the hyperplane, thus the length of the projection of $v_{n+1} - v_i$ onto n is $h - 0 = h$ still.

From this, with appropriate switching of rows, we can subtract a linear combination L of $v_1, \dots, v_n$ to v_{n+1} such that $v_{n+1} - L$ is a vector of all 0's except for a single component. This component must be h since the length of the projection is less than or equal to the length of the vector.

$$\det[v_1 \dots v_{n+1}] = \det[v_1 \dots, v_n, (v_{n+1} - L)] = \det \begin{bmatrix} | & \dots & | & 0 \\ v_1 & \dots & v_n & \vdots \\ | & \dots & | & 0 \\ | & \dots & | & h \end{bmatrix}$$

Now, we observe a similar argument: adding a linear combination of $v_{n+1}-L = v'_{n+1}$ to v_n will not change the volumes, because v'_{n+1}'s remaining entry must be in the direction of n, and hence adding v'_{n+1} to all the v_i is just a translation of the hyperplane.

We can use this to subtract a multiple of v'_{n+1} such that the last entry of $v'_i = v_i - K_i v'_{n+1}$ is 0

$$\det[(v_1 - K_1 v'_{n+1}), \dots (v_n - K_n v'_{n+1}), (v_{n+1} - L)] = \det \begin{bmatrix} | & \dots & | & 0 \\ v'_1 & \dots & v'_n & \vdots \\ | & \dots & | & 0 \\ 0 & \dots & 0 & h \end{bmatrix}$$

Finally, note that $\{v_1, \dots, v_n\}$ and $\{v'_1, \dots, v'_n\}$ yield parallelopipids of the same volume since they are translates of each other. Now, we evaluate the above by expanding along the last column and have:

$$h \det \begin{bmatrix} | & \dots & | \\ v'_1 & \dots & v'_n \\ | & \dots & | \end{bmatrix} = h \times \mathrm{Vol}(v_1, \dots v_n) = \mathrm{Vol}(v_1, \dots v_{n+1})$$

Therefore, by induction, the determinant of the vectors yields the volume of the parallelepiped.

1.5.3 $\ker S$ is T-invariant.

Say $x \in \ker T$, then $S(x) = 0$.
$0 = T(0) = T \circ S(x) = S \circ T(x)$, so $T(x) \in \ker S$.

$S(V)$ is T-invariant.
say $x \in S(V)$, then there exists a v such that $S(v) = x$
$T(x) = T \circ S(v) = S \circ T(v)$, so $T(x) \in S(V)$.

1.6.4 Cayley-Hamilton implies that there exists $f(z)$ such that $f(T) = 0$ of the form $f(z) = \prod_{i=1}^{m}(z - \lambda_i)$. If you distribute it out, you will get the following polynomial:

$$f(z) = z^n + (-\lambda_1 - \cdots - \lambda_n)z^{n-1} + \cdots + cz + (-\lambda_1)(\ldots)(-\lambda_n)$$

Notice the product of eigenvalues is the determinant and the sum of eigenvalues is the trace. To simplify, we write them as such.
Plugging in T and rearranging gives the following:

$$0 = f(T) = T^n - \operatorname{Tr}(T)T^{n-1} + + \cdots + cT + (-1)^n(\det T)I$$

$$I = \frac{(-1)^{n+1}}{\det T}(T^n - \operatorname{Tr}(T)T^{n-1} + \cdots + cT)$$

$$I = T\left(\frac{(-1)^{n+1}}{\det T}T^{n-1} + \frac{(-1)^{n+2}\operatorname{Tr} T}{\det T}T^{n-2} + \cdots + \frac{(-1)^{n+1}c}{\det T}\right)$$

$$\left(\frac{(-1)^{n+1}}{\det T}T^{n-1} + \frac{(-1)^{n+2}\operatorname{Tr} T}{\det T}T^{n-2} + \cdots + \frac{(-1)^{n+1}c}{\det T}\right) = T^{-1}$$

$$\text{Let } P(z) = \left(\frac{(-1)^{n+1}}{\prod_{i=1}^{m}\lambda_i}z^{n-1} + \frac{(-1)^{n+2}\sum_{i=1}^{m}\lambda_i}{\prod_{i=1}^{m}\lambda_i}z^{n-2} + \cdots + \frac{(-1)^{n+1}c}{\prod_{i=1}^{m}\lambda_i}\right)$$

Then $P(z)$ is a polynomial such that $P(T) = T^{-1}$

1.6.5 Let k and h be the exponents on $(z - 4)$ and $(z - 5)$ in f(z), characteristic polynomial of T. Then $f(T) = (T - 4I_v)^k(T - 5I_v)^h = 0$.

Since V is n-dimensional, it can only have n eigenvalues counting multiplicity. Since 4 and 5 are the only eigenvalues, we must have that $h \leq n - 1$ and $k \leq n - 1$, so the characteristic polynomial divides $(z - 4)^{n-1}(z - 5)^{n-1}$. Therefore $(T - 4I_v)^{n-1}(T - 5I_v)^{n-1} = f(T)q(T) = 0q(T) = 0$.

1.7.3 Let $v \in V$. For any $v' \in V$, $v = (v - T^n v') + T^n v'$.
Since $Tv' \in T^n(V)$ for all v', we need only find a v' such that $v - T^n v' \in \ker T^n$.
For this to be true, we must have $T^n(v - T^n v') = T^n v - T^{2n} v' = 0$.
Equivalently, we need a v' such that $T^n v = T^{2n} v'$.
Since $n = \dim V$, $T^n(V) = T^{2n}(V)$, so there must be such a v'.

The sum has to be a direct sum due to exercise 1.1.4 and rank-nullity.

1.7.5 First we note that if $\ker T^{i-1} \subsetneq \ker T^i$, then there exists a vector v such that $T^{i-1} v \neq 0$, but $T^i v = 0$, so the dimension of $\ker T^i$ is at least one more than $\ker T^{i-1}$. Since this holds for the entire chain of eigenspaces, we must have that the dimension of $\ker T^{n-1}$ is $(n-1)$.

That means that there are $(n-1)$ linearly independant vectors such that $T^{n-1} v = 0$, i.e. the eigenvalue $\lambda = 0$ has multiplicity at least $(n-1)$. Since the dimension of V is n, there can only be one more eigenvalue, so T has at most two distinct eigenvalues.

1.8.1 $\Rightarrow$
If N is nilpotent, then there exists an m such that $N^m = 0$. Say λ is an eigenvalue of N. It must have an eigenvector $v \neq 0$ satisfying $Nv = \lambda v$.
Applying N m times gives the following:

$$0 = 0v = N^m v = N^{n-1} N v = N^{n-1} \lambda v = N^{n-2} N \lambda v = N^{n-2} \lambda^2 v = \cdots = \lambda^m v$$

So we have $\lambda^m = 0$, hence $\lambda = 0$.

$\Leftarrow$
Since the only eigenvalue of N is 0, then the characteristic polynomial of N is $p(z) = z^n$. By Cayley-Hamilton theorem, $p(N) = N^n = 0$, making N nilpotent.

1.9.4 Characteristic Polynomial: $(x - \lambda_1)^3 (x - \lambda_1)^3$
Minimal Polynomial: $(x - \lambda_1)^3 (x - \lambda_1)^2$

1.9.5 Let $r' = \min\{r : (T - \lambda_i I_v)^r|_{V_i} = 0\}$.
Say $r' > r_i$

Then $\ker(T - \lambda_i I_v)^{r_i} \subsetneq \ker(T - \lambda_i I_v)^{r'}$

So there exists $v \in V_i$, $v \neq 0$ such that $(T - \lambda_i I_v)^{r_i} v \neq 0$.

Recall that $0 = P_{\min}(T) = \prod (T - \lambda_i)^{r_i}$, so it must be that $\prod_{j \neq i} (T - \lambda_j)^{r_j} v = 0$.

This implies that v has to be in some second generalized eigenspace. V is the direct product of the generalized eigenspaces, so v cannot be contained in some other eigenspace without being the zero vector, so we have a contradiction.

Now say $r' < r_i$
$(T - \lambda_i I_v)^{r'} = 0$. So $(z - \lambda_i)^{r'}$ divides $P_{\min}(z)$.
This contradicts $P_{\min}(z)$ being minimal.

Therefore $r' = r_i$.

1.9.6 Since V is the direct sum of generalized eigenspace, $v \in \ker(T - \lambda I_V)^m$ for some eigenvalue λ. This means that there exists a $l \leq m$ such that $(T - \lambda I_V)^l v = 0$, but $(T - \lambda I_V)^{l-1} v \neq 0$.

Define $s(z) = (z - \lambda)^l$. Then $s(T)v = 0$ and $s(z)$ is monic. Any monic polynomial is a product of linear factors. Since generalized eigenspaces only trivially intersect, there is no other linear factors other than $(z - \lambda)$ to which $s(T)v = 0$ is possible. Since l is the smallest exponent for which v is in the kernel, so it is of lowest degree.

Let $P_{\min}(z)$ be the minimal polynomial of T. By the divison algorithm, there exists $q(z)$ and $r(z)$ such that $P_{\min}(z) = s(z)q(z) + r(z)$ with degree of $r(z)$ strictly less than the degree of $s(z)$.
Since $P_{\min}(T)v = 0$ and $s(T)v = 0$, we must have $r(T)v = 0$.
If $r(z) \neq 0$, then this contradicts $s(z)$ being of smallest degree.

Thus $P_{\min}(z) = s(z)q(z)$, so $s(z)$ divides the minimal polynomial.

1.9.7 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1.9.12 This can be proved by induction. The base case is Prop 1.9.11.
Say there exists a basis for which $S_1, \ldots, S_k$ are diagonal.
Then in that basis, $S' = S_1 S_2 \ldots S_k$ is a product of diagonal matrices and thus must be diagonal, hence S' is a diagonalizable operator.
Since the S_i's all pairwise commute, we can notice the following:

$$S_{k+1}S' = \underline{S_{k+1}S_1}S_2 \ldots S_k = S_1\underline{S_2S_{k+1}} \ldots S_k = \cdots = S_1S_2 \ldots \underline{S_kS_{k+1}} = S'S_{k+1}$$

So S_{k+1} and S' are diagonalizable linear operators, so there must be a basis that diagonalizes both of them by Prop 1.9.11.
Since diagonalization is unique, this diagonalization of S' from this basis must be the same as the one from our hypothesis, which means that this basis must also diagonalize S_1 through S_k.

1.10.3 Let $B = \{b_1, \ldots, b_n\}$ be a basis for V.
Then $T(b_j) = \sum_{i=1}^n a_{ij} b_i$ and $v = \sum_{j=1}^n v_j b_j$
So the (ij)th component of $M_{B,B}(T)$ is a_{ij} and the jth component of $[v]_B$ is v_j.
This means the ith component of $M_{B,B}(T)[v]_B = \sum_{j=1}^n a_{ij} v_j$

$$T(v) = T\left(\sum_{j=1}^n v_j b_j\right) = \sum_{j=1}^n v_j T(b_j) = \sum_{j=1}^n \sum_{i=1}^n v_j a_{ij} b_i = \sum_{i=1}^n \left(\sum_{j=1}^n v_j a_{ij}\right) b_i$$

Then the ith component of $[Tv]_B$ is $\sum_{j=1}^n v_j a_{ij}$.

Since they are equal componentwise, $[Tv]_B = M_{B,B}(T)[v]_B$.

Chapter 2: Lie Algebras: Definition and Basic Properties

2.1.5 $\overline{a} \times (\overline{b} \times \overline{c}) = \overline{a} \times ((b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k})$

The ith component of which is $(a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))$

$= a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 = b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3c_3)$

add and subtract $a_1b_1c_1$ and we have

$b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) = b_1(\overline{a} \cdot \overline{c}) - c_1(\overline{a} \cdot \overline{b})$

Similarly, the jth component is $(a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1))$

$a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 = b_2(a_1c_1 + a_3c_3) - c_2(a_1b_1 + a_3c_3)$

add and subtract $a_2b_2c_2$ and we have

$b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3c_3) = b_2(\overline{a} \cdot \overline{c}) - c_2(\overline{a} \cdot \overline{b})$

And thirdly, the k component is $(a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))$

$a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 = b_3(a_1c_1 + a_2c_2) - c_3(a_1b_1 + a_2b_2)$

add and subtract $a_3b_3c_3$ and we have

$b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3c_3) = b_3(\overline{a} \cdot \overline{c}) - c_3(\overline{a} \cdot \overline{b})$

Thus we have $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$

If we switch letters around, we have: $\overline{b} \times (\overline{c} \times \overline{a}) = (\overline{b} \cdot \overline{a})\overline{c} - (\overline{b} \cdot \overline{c})\overline{a}$

and lastly $\overline{c} \times (\overline{a} \times \overline{b}) = (\overline{c} \cdot \overline{b})\overline{a} - (\overline{c} \cdot \overline{a})\overline{b}$

$$\overline{a} \times (\overline{b} \times \overline{c}) + \overline{b} \times (\overline{c} \times \overline{a}) + \overline{c} \times (\overline{a} \times \overline{b})$$
$$= (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c} + (\overline{b} \cdot \overline{a})\overline{c} - (\overline{b} \cdot \overline{c})\overline{a} + (\overline{c} \cdot \overline{b})\overline{a} - (\overline{c} \cdot \overline{a})\overline{b}$$

$$= \cancel{(\overline{a} \cdot \overline{c})\overline{b} - (\overline{c} \cdot \overline{a})\overline{b}} \cancel{- (\overline{a} \cdot \overline{b})\overline{c} + (\overline{b} \cdot \overline{a})\overline{c}} \cancel{- (\overline{b} \cdot \overline{c})\overline{a} + (\overline{c} \cdot \overline{b})\overline{a}} = 0$$

Thus the cross product satisfies the Jacobi identity.

$\overline{a} \times \overline{a} = (a_2a_3 - a_3a_2)\mathbf{i} + (a_3a_1 - a_1a_3)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} = 0$, so it is alternating.

$$(\alpha\overline{a}+\beta\overline{c})\times\overline{c} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha a_1+\beta c_1 & \alpha a_2+\beta c_2 & \alpha a_3+\beta c_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= ((\alpha a_2+\beta c_2)b_3 - (\alpha a_3+\beta c_3)b_2)\mathbf{i} + ((\alpha a_3+\beta c_3)b_1$$
$$-(\alpha a_1+\beta c_1)b_3)\mathbf{j} + ((\alpha a_1+\beta c_1)b_2 - (\alpha a_2+\beta c_2)b_1)\mathbf{k}$$

$$= \alpha(a_2b_3 - a_3b_2)\mathbf{i} + \alpha(a_3b_1 - a_1b_3)\mathbf{j} + \alpha(a_1b_2 - a_2b_1)\mathbf{k}$$
$$+\beta(c_2b_3 - c_3b_2)\mathbf{i} + \beta(c_3b_1 - c_1b_3)\mathbf{j} + \beta(c_1b_2 - c_2b_1)\mathbf{k}$$

$$= \alpha\overline{a}\times\overline{c} + \beta\overline{b}\times Y$$

And thus the cross product is linear.

Therefore it $(\mathbb{R}^3, \times)$ is a Lie algebra.

2.1.9 $(\mathcal{D}_1+\mathcal{D}_2)(a.b) = \mathcal{D}_1(a.b)+\mathcal{D}_2(a.b) = \mathcal{D}_1(a).b + a.\mathcal{D}_1(b) + \mathcal{D}_2(a).b + a.\mathcal{D}_2(b)$
$= \mathcal{D}_1(a).b + \mathcal{D}_2(a).b + a.\mathcal{D}_1(b) + a.\mathcal{D}_2(b) = (\mathcal{D}_1(a)+\mathcal{D}_2(a)).b + a.(\mathcal{D}_1(b)+\mathcal{D}_2(b))$
$= ((\mathcal{D}_1+\mathcal{D}_2)(a)).b + a.((\mathcal{D}_1+\mathcal{D}_2)(b))$, thus $(\mathcal{D}_1+\mathcal{D}_2) \in \text{Der}\,\mathcal{A}$

$\lambda\mathcal{D}(a.b) = \lambda(\mathcal{D}(a).b + a.\mathcal{D}(b)) = \lambda\mathcal{D}(a).b + \lambda a.\mathcal{D}(b) = (\lambda\mathcal{D})(a).b + a.(\lambda\mathcal{D})(b)$
Thus, $\lambda\mathcal{D} \in \text{Der}\,\mathcal{A}$.

2.1.20 Consider the $2n\times 2n$ matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

$X \in sp(n,\mathbb{F})$ iff $J_n({}^tX)J_n^{-1} = -X$ holds.
Note that $J_n^{-1} = -J_n = {}^tJ_n$ and observe:

$$J_n({}^tX)J_n^{-1} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{bmatrix}\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} {}^tB & {}^tD \\ -{}^tA & -{}^tC \end{bmatrix}\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

$$= \begin{bmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{bmatrix} = \begin{bmatrix} -A & -B \\ -C & -D \end{bmatrix} = -X$$

We get $D = -{}^tA$, $B = {}^tB$, and $C = {}^tC$.

2.1.25 For $\mathfrak{sl}(n,\mathbb{F})$, $\mathrm{Tr}({}^tX) = \mathrm{Tr}(X) = 0$,
Thus ${}^tX$ satisfies the condition, hence $\mathfrak{sl}(n,\mathbb{F})$ is invariant.

For $\mathfrak{so}(n,\mathbb{F})$, ${}^tX = -X$, applying ^{t} to X gives
${}^t({}^tX) = {}^t(-X) = -({}^tX)$, so it is invariant.
Thus ${}^tX$ satisfies the condition, hence $\mathfrak{so}(n,\mathbb{F})$ is invariant.

For $\mathfrak{sp}(n,\mathbb{F})$, $J_n({}^tX)J_n^{-1} = -X$, applying ^{t} to X gives:

$$J_n({}^t({}^tX))J_n^{-1} = J_n({}^t({}^tX))({}^tJ_n) = {}^t(J_n({}^tX))({}^tJ_n)) = {}^t(J_n({}^tX)J_n^{-1}) = {}^t(-X) = -{}^tX$$

Thus ${}^tX$ satisfies the condition, hence $\mathfrak{sp}(n,\mathbb{F})$ is invariant.

For $\mathfrak{u}(n)$, $(X^*)^* = (-X)^* = -X^*$.
Thus X^* satisfies the condition, hence $\mathfrak{u}(n)$ is invariant.

For $\mathfrak{su}(n,\mathbb{F})$, $\mathrm{Tr}(X^*) = \mathrm{Tr}(X) = 0$. Combined with the above, we have that X^* satisfies the condition, hence $\mathfrak{su}(n)$ is invariant.

For $\mathfrak{u}(p,q)$, $I_{p,q}(X^*)I_{p,q} = -X$, note $I_{p,q}^* = I_{p,q}$.

$$I_{p,q}((X^*)^*)I_{p,q} = I_{p,q}((X^*)^*)I_{p,q}^* = (I_{p,q}((X^*)^*)I_{p,q}^*)^* = (I_{p,q}((X^*)I_{p,q})^* = (-X)^* = -X^*$$

Thus X^* satisfies the condition, hence $\mathfrak{u}(p,q)$ is invariant.

For $\mathfrak{su}(p,q)$, the above condition combined with $\mathrm{Tr}(X^*) = \mathrm{Tr}(X) = 0$ gives that $\mathfrak{su}(p,q)$ is invariant.

Chapter 3: Basic Algebraic Facts

3.2.2 Say $\mathfrak{g}$ is not abelian. Let $\{A, B\}$ be a basis for $\mathfrak{g}$.
$[A, B] = C = aA + bB$.

$$[A, C] = [A, aA + bB] = a\cancel{[A, A]} + b[A, B] = b(aA + bB) = bC$$

Consider $X = \frac{1}{b}A$ and $Y = \frac{1}{b}C$.

$$[X, Y] = \left[\frac{1}{b}A, \frac{1}{b}C\right] = \frac{1}{b^2}[A, C] = \frac{1}{b^2}bC = \frac{1}{b}C = Y$$

Hence, we have a basis $\{X, Y\}$ of $\mathfrak{g}$ such that $[X, Y] = Y$.

3.2.3 Elements of $\mathfrak{so}(3)$ are of the form $\begin{bmatrix} 0 & a_1 & a_3 \\ -a_1 & 0 & a_2 \\ -a_3 & -a_2 & 0 \end{bmatrix}$, so we could consider the map ϕ that sends such elements to (a_1, a_2, a_3). Clearly it is bijective.

$$\begin{bmatrix} 0 & a_1 & a_3 \\ -a_1 & 0 & a_2 \\ -a_3 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_1 & b_3 \\ -b_1 & 0 & b_2 \\ -b_3 & -b_2 & 0 \end{bmatrix} = \begin{bmatrix} -a_1b_1 - a_3b_3 & -a_3b_2 & a_1b_2 \\ -a_2b_3 & -a_1b_1 - a_2b_2 & -a_1b_3 \\ a_2b_1 & -a_3b_1 & -a_3b_3 - a_2b_2 \end{bmatrix}$$

$$\begin{bmatrix} -a_1b_1 - a_3b_3 & -a_3b_2 & a_1b_2 \\ -a_2b_3 & -a_1b_1 - a_2b_2 & -a_1b_3 \\ a_2b_1 & -a_3b_1 & -a_3b_3 - a_2b_2 \end{bmatrix} - \begin{bmatrix} -b_1a_1 - b_3a_3 & -b_3a_2 & b_1a_2 \\ b_2a_3 & -b_1a_1 - b_2a_2 & -b_1a_3 \\ b_2a_1 & -b_3a_1 & -b_3a_3 - b_2a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_2b_3 - a_3b_2 & a_1b_2 - b_1a_2 \\ -(b_3 - a_3b_2) & 0 & a_3b_1 - a_1b_3 \\ -(a_1b_2 - b_1a_2) & -(a_3b_1 - a_1b_3) & 0 \end{bmatrix} = [X, Y]$$

Under ϕ, this maps exactly to $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$, therefore we have $\phi([X, Y]) = [\phi(X), \phi(Y)]$, and therefore ϕ is a Lie algebra isomorphism.

3.2.8 Let $g \in \mathfrak{g}$. Clearly $g = \frac{g+\varphi(g)}{2} + \frac{g-\varphi(g)}{2}$

$$\varphi\left(\frac{g+\varphi(g)}{2}\right) = \frac{1}{2}(\varphi(g) + \varphi(\varphi(g))) = \frac{1}{2}(\varphi(g) + g)) = \frac{g+\varphi(g)}{2}$$

$$\varphi\left(\frac{g-\varphi(g)}{2}\right) = \frac{1}{2}(\varphi(g) - \varphi(\varphi(g))) = \frac{1}{2}(\varphi(g) - g)) = -\frac{g+\varphi(g)}{2}$$

Thus $\frac{g+\varphi(g)}{2} \in \mathfrak{h}$ and $\frac{g-\varphi(g)}{2} \in \mathfrak{q}$ and we have a decomposition, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$.
Say $h \in \mathfrak{h} \cap \mathfrak{q}$, then $h = \phi(h) = -h$, so $h = 0$, thus $\mathfrak{h} \cap \mathfrak{q} = \{0\}$, making our sum direct as per 1.1.2, so $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$.

$$\varphi([h,h]) = [\varphi(h), \varphi(h)] = [h,h] \text{ , thus } [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$$
$$\varphi([q,q]) = [\varphi(q), \varphi(q)] = [-q,-q] = [q,q] \text{ , thus } [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$$
$$\varphi([h,q]) = [\varphi(h), \varphi(q)] = [h,-q] = -[h,q] \text{ , thus } [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$$

3.4.7 $\Rightarrow$

Say $\mathfrak{g}$ is an ideal of $\mathfrak{m}$. We must show $\mathfrak{m} = \mathfrak{g} \bigoplus \mathfrak{h}$ for some $\mathfrak{h}$.
Define $\mathfrak{h} = \mathfrak{g}^{\perp} := \{m \in \mathfrak{m} : [m,g] = 0, \forall g \in \mathfrak{g}\}$

For all $z \in \mathfrak{m}$, $\mathrm{ad}(z)|_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$.
Since it is complete, this implies that $\mathrm{ad}(z)|_{\mathfrak{g}} = \mathrm{ad}(x)$ for some $x \in \mathfrak{g}$.
$\mathrm{ad}(z-x)|_{\mathfrak{g}} = 0$, so $z - x \in \mathfrak{h}$. Let $y = z - x$,
then we have for all $z \in \mathfrak{m}$, $z = x + y$ where $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$.
Therefore $\mathfrak{m} = \mathfrak{g} + \mathfrak{h}$.

Say $x \in \mathfrak{g} \cap \mathfrak{h}$, then, $[x,g] = 0$ for all g, and $x \in \mathfrak{g}$. This means $x \in \mathfrak{c}$. Since $\mathfrak{c} = \{0\}$, $\mathfrak{g} \cap \mathfrak{h} = \{0\}$, so the sum is direct.

$\Leftarrow$

To show it is complete, we must show $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.
We always have that $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$ is an ideal.
Note that $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}$, so we have that it is an ideal of $\mathrm{Der}(\mathfrak{g})$, and hence $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \bigoplus \mathfrak{h}$ for some $\mathfrak{h}$ by hypothesis.

Say $D \in \mathfrak{h}$. Then $0 = [\mathrm{ad}X, D] = \mathrm{ad}(Dx)$. Then Dx commutes with everything, hence $Dx = 0$ for all x since $\mathfrak{c} = \{0\}$, which implies $D = 0$.

Thus $\mathfrak{h} = \{0\}$, therefore $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

3.4.8 This question is difficult. If you have an idea on how to solve it, please send me an email.

Chapter 4: Solvable Lie Algebras and Lies Theorem

4.1.8 $\Rightarrow$

The derived sequence from definition 4.1.5 is such a sequence, and the abelian condition holds by Theorem 4.1.2.

$\Leftarrow$

From Theorem 4.1.2, $\mathfrak{g}/\mathfrak{g}_1$ is abelian implies $\mathfrak{g}^{(1)} \subset \mathfrak{g}_1$.
Similarly, $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian implies $\mathfrak{g}^{(i+1)} \subset \mathfrak{g}_{i+1}$ for all i.
Thus $\mathfrak{g}_{m-1}/\mathfrak{g}_m$ being abelian implies $\mathfrak{g}^{(m)} \subset \mathfrak{g}_m \subset \{0\}$, so $\mathfrak{g}$ is solvable.

4.1.15 Call the map f. The image of f is by definition $U' + U''$.
Since the intersection is $\{0\}$, by Exercise 1.1.2, the sum is direct.
Thus all that remains to be shown is that the map is linear.

$$f(k(u', u'') + (v', v'')) = f((ku' + v', ku'' + v'') = ku' + v' + ku'' + v''$$
$$= k(u' + u'') + v' + v'' = k(f(u', u'') + f(v', v'')$$

4.1.16 Linearity first slot:

$$[c(x_1, y_1) + (x_3, y_3), (x_2, y_2)] = [(cx_1 + x_3, cy_1 + y_3), (x_2, y_2)]$$
$$= ([cx_1 + x_3, x_2]_{\mathfrak{g}}, [cy_1 + y_3, y_2]_{\mathfrak{h}}) = (c[x_1, x_2]_{\mathfrak{g}} + [x_3, x_2]_{\mathfrak{g}}, c[y_1, y_2]_{\mathfrak{h}} + [y_3, y_2]_{\mathfrak{h}})$$
$$= c([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{h}}) + ([x_3, x_2]_{\mathfrak{g}}, [y_3, y_2]_{\mathfrak{h}})$$

Linearity in the second slot is similar.

Anticommutative:

$$[(x_1, y_1), (x_1, y_1)] = ([x_1, x_1]_{\mathfrak{g}}, [y_1, y_1]_{\mathfrak{h}}) = (0, 0)$$

Jacobi identity: We seek to prove the following:

$$[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] + [(x_2, y_2), [(x_3, y_3), (x_1, y_1)]] + [(x_3, y_3), [(x_1, y_1), (x_2, y_2)] = 0$$

To do this, we look at the first expression and notice the following:

$$[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] = [(x_1, y_1), ([x_2, x_3]_{\mathfrak{g}}, [y_2, y_3]_{\mathfrak{h}})] = ([x_1, [x_2, x_3]_{\mathfrak{g}}]_{\mathfrak{g}}, [y_1, [y_2, y_3]_{\mathfrak{h}}]_{\mathfrak{h}})$$

So if we compute the other two similarly, when we add them together then our first slot is $([x_1, [x_2, x_3]_\mathfrak{g}]_\mathfrak{g} + [x_3, [x_1, x_2]_\mathfrak{g}]_\mathfrak{g} + [x_2, [x_3, x_1]_\mathfrak{g}]_\mathfrak{g}$, which must be 0 since it is the Jacobi identity on $\mathfrak{g}$.

By the same argument, $([y_1, [y_2, y_3]_\mathfrak{h}]_\mathfrak{h} + [y_3, [y_1, y_2]_\mathfrak{h}]_\mathfrak{h} + [y_2, [y_3, y_1]_\mathfrak{h}]_\mathfrak{h}$ is the second slot, which must be 0 since it is the Jacobi identity on $\mathfrak{h}$.

4.1.17 The Lie bracket on $\mathfrak{g}$ is is the Lie bracket of $\mathfrak{sl}(2, \mathbb{C})$ on each of the components, so an ideal in $\mathfrak{g}$ must be the product of two ideals in $\mathfrak{sl}(2, \mathbb{C})$. However, by example 3.3.7, $\mathfrak{sl}(2, \mathbb{C})$ is simple, and thus has no nonzero ideals.
Therefore, the only ideals of $\mathfrak{g}$ are $\mathfrak{g}, \{(0, 0)\}, \mathfrak{sl}(2, \mathbb{C}) \times \{0\}$ and $\{0\} \times \mathfrak{sl}(2, \mathbb{C})$.

Consider the ideal $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C}) \times \{0\} \subset \mathfrak{g}$.
We can check that it is indeed an ideal by the following: For any $(x_1, y_1) \in \mathfrak{g}$ and $(x_2, 0) \in \mathfrak{h}$,, we have $[(x_1, y_1), (x_2, 0)] = ([x_1, x_2], [y_1, 0]) = ([x_1, x_2], 0) \in \mathfrak{h}$.
$\mathfrak{h}$ is a nonzero ideal of $\mathfrak{g}$, so $\mathfrak{g}$ cannot be simple.

Since $\mathfrak{sl}(2, \mathbb{C})$ is simple, it is not solvable as per the remark on the bottom of page 60, hence $\mathfrak{sl}(2, \mathbb{C}) \times \{0\}$ and $\{0\} \times \mathfrak{sl}(2, \mathbb{C})$ cannot be solvable, making $\{(0, 0)\}$ the only solvable ideal of $\mathfrak{g}$. This means the radical of $\mathfrak{g}$ must be the zero ideal, making $\mathfrak{g}$ semisimple.

4.1.19 Say $\mathfrak{h} \subset \mathfrak{g}$ is a semisimple subalgebra. By the observations on the bottom of page 62, any subalgebra of a solvable lie algebra is solvable, so $\mathfrak{h}$ is solvable. The definition of semisimple is that the maximal solvable ideal of $\mathfrak{h}$ is $\{0\}$. Thus the only way that $\mathfrak{h}$ can be solvable is if it is contained in $\{0\}$, so $\mathfrak{h} = \{0\}$.

Chapter 5: Nilpotent Lie Algebras and Engels Theorem

5.1.5 (⇒)
From being nilpotent, the descending central series suffices.

(⇐)
First we note that $\mathfrak{g}_0 = \mathcal{C}^0$ and say $\mathcal{C}^i \subset \mathfrak{g}_i$ for induction.
$\mathcal{C}^{i+1} = [\mathcal{C}^i, \mathfrak{g}] \subset [\mathfrak{g}_i, \mathfrak{g}]$
Since $\mathfrak{g}_i/\mathfrak{g}_{i+1} \subset \mathfrak{c}(\mathfrak{g}/\mathfrak{g}_{i+1})$, $[\mathfrak{g}_i/\mathfrak{g}_{i+1}, \mathfrak{g}/\mathfrak{g}_{i+1}] = 0$, meaning $[\mathfrak{g}_i, \mathfrak{g}] \subset \mathfrak{g}_{i+1}$

Then we have $\mathcal{C}^{i+1} \subset \mathfrak{g}_{i+1}$.

If $\mathfrak{g}_m = \{0\}$ for some m, it must be that $\mathcal{C}^m = \{0\}$, hence it is nilpotent.

5.1.6 First we prove $\mathfrak{g}^{(i)} \subset \mathcal{C}^i(\mathfrak{g})$ for all i.

The base case is $\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}^{(1)}$

Assume $\mathcal{C}^i(\mathfrak{g}) \supset \mathfrak{g}^{(i)}$ for induction.

$\mathfrak{g}^{(i)} \subset \mathfrak{g}$ and $\mathfrak{g}^{(i)} \subset \mathcal{C}^i(\mathfrak{g}) \Rightarrow [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \subset [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}]$.

By definition, $\mathcal{C}^{i+1}(\mathfrak{g}) = [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}]$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

Thus we have $\mathfrak{g}^{(i+1)} \subset \mathcal{C}^{i+1}(\mathfrak{g})$ as desired.

If $\mathfrak{g}$ is nilpotent, then for some m, $\mathcal{C}^m(\mathfrak{g}) = \{0\}$.
Since $\mathfrak{g}^{(m)} \subset \mathcal{C}^m(\mathfrak{g})$, we have $\mathfrak{g}^{(m)} = \{0\}$, making it solvable.

5.1.8 $T_n(\mathbb{F})$
First note that the identity matrix is an upper triangular matrix and it commutes with everything in $T_n(\mathbb{F})$. Moreover, scalar multiples of the identity (scalar matrices) also commute with everything in $T_n(\mathbb{F})$. We show this is all.
Say $X \in \mathfrak{c}(T_n(\mathbb{F}))$.
Consider the matrix Λ consisting of all zeros except for two entries along the diagonal, $\lambda_i \neq \lambda_j$ in the (i, i) and (j, j) slots respectively, $i < j$.

The (i, j)th entry of $[X, \Lambda]$ is $\lambda_i X_{ij} - \lambda_j X_{ij}$.
This, however, must be zero, so then $X_{ij} = 0$.

This means that X has only nozero entries along the diagonal. To prove they are all equal, consider that X must commute with the matrices E_{ij}, which has a 1 in the (i, j) slot.

The (i, j)th entry of $[X, E_{ij}]$ is $X_{ii} - X_{jj}$.
This, however, must be zero, so then $X_{ii} = X_{jj}$.

Therefore X is a scalar matrix. The space of scalar matrices is of dimension 1, so then $T_n(\mathbb{F})$ has center of dimension 1.

$U_n(\mathbb{F})$
First note that matrices whose only entry is the upper right corner commute with everything in $U_n(\mathbb{F})$, as multiplying one with any strictly upper triangular matrix yields the zero matrix. We show this is all.
Say $X \in \mathfrak{c}(U_n(\mathbb{F}))$.
$[X, E_{1n}] = 0$, obviously, so consider E_{ij} for $i < j$ except $(1, n)$.
The (i, j)th entry of $[X, E_{ij}]$ is $X_{ij} - \cancel{X_{ji}} = X_{ij}$ since X is upper triangular.
Hence $X_{ij} = 0$ for $i < j$ except $(1, n)$.
Then the only element of X that is nonzero is the $(1, n)$th entry, so the center consits of the space of matrices with a single nonzero entry in the top right corner, which is clearly of dimension 1.

5.1.9 Multiplying any two elemens leaves only the top right corner nonzero. i.e.

$$\begin{bmatrix} 0 & t_1 & \dots & t_n & c \\ & & & & w_1 \\ & & \mathbf{0} & & \vdots \\ & & & & w_n \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & t'_1 & \dots & t'_n & c' \\ & & & & w'_1 \\ & & \mathbf{0} & & \vdots \\ & & & & w'_n \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & cw'_1 \\ & & & & 0 \\ & & \mathbf{0} & & \vdots \\ & & & & 0 \\ & & & & 0 \end{bmatrix}$$

Clearly then, taking the Lie bracket leaves matrices with nonzero entries in the top right corner. Then $\mathfrak{g}_1$ is the one-dimensional Lie algebra.

Furthermore, multiplying a matrix with only an entry in the top right corner yields the zero matrix as the first column and last row are all zero, so there is never a nonzero entry. This makes the Lie bracket of any matrix from $\mathfrak{g}_1$ with anything in $\mathfrak{h}_n$ zero as below.

$$\begin{bmatrix} 0 & t_1 & \dots & t_n & c \\ & & & & w_1 \\ & & \mathbf{0} & & \vdots \\ & & & & w_n \\ & & & & 0 \end{bmatrix} \begin{bmatrix} & & x \\ & \mathbf{0} & \\ & & \end{bmatrix} - \begin{bmatrix} & & x \\ & \mathbf{0} & \\ & & \end{bmatrix} \begin{bmatrix} 0 & t_1 & \dots & t_n & c \\ & & & & w_1 \\ & & \mathbf{0} & & \vdots \\ & & & & w_n \\ & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix} - \begin{bmatrix} 0 \end{bmatrix}$$

The descending central series is $\mathfrak{h}_n \supset \mathbb{R} \supset \{0\}$, making the general Heisenberg Lie algebra 2-step nilpotent.

5.1.10 As per exercise 3.2.2,we know that if $\mathfrak{g}$ is a 2-dimensional lie algebra, then it has basis $\{X, Y\}$ such that $[X, Y] = Y$.
Let $\mathfrak{h}$ be the 1-dimensional Lie algebra generated by Y.

$$[aX+bY, a'X+b'Y] = aa'\cancel{[X,X]}+ab'[X,Y]+ba'[Y,X]+bb'\cancel{[Y,Y]} = (ab'-a'b)Y \in \mathfrak{h}$$

Clearly then, $\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}^1 = \mathfrak{h}$

$$[aX + bY, cY] = ac[X, Y] + bc\cancel{[Y,Y]} = acY \in \mathfrak{h}$$

Hence $\mathcal{C}^2(\mathfrak{g}) = \mathfrak{h}$. By the same argument, $\mathcal{C}^i(\mathfrak{g}) = \mathfrak{h}$ for all $i > 1$.

However, $[aY, bY] = 0$, so we have $\mathfrak{g}^2 = \{0\}$.

Therefore the derived series is:

$$\mathfrak{g} \supset \mathfrak{h} \supset \{0\}$$

Making $\mathfrak{g}$ solvable.

The Descending central series is:

$$\mathfrak{g} \supset \mathfrak{h} \supset \mathfrak{h} \supset \mathfrak{h} \supset \dots$$

Making $\mathfrak{g}$ not nilpotent.

5.1.11 1)
Since $\mathfrak{h} \subset \mathfrak{g}$, clearly $\mathcal{C}^1(\mathfrak{h}) \subset \mathcal{C}^1(\mathfrak{g})$ /
For induction, say $\mathcal{C}^i(\mathfrak{h}) \subset \mathcal{C}^i(\mathfrak{g})$.
$\mathcal{C}^{i+1}(\mathfrak{h}) = [\mathcal{C}^i(\mathfrak{h}), \mathfrak{h}] \subset [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}] = \mathcal{C}^{i+1}(\mathfrak{g})$

So $\mathcal{C}^i(\mathfrak{h}) \subset \mathcal{C}^i(\mathfrak{g})$ for all i.

If $\mathfrak{g}$ were nilpotent, then $\mathcal{C}^m(\mathfrak{g}) = \{0\}$ for some m.
By above, $\mathcal{C}^i(\mathfrak{h}) = \{0\}$, making $\mathfrak{h}$ nilpotent as well.

2)
By Prop 5.1.4, we need only look at the ascending central series.
If $\mathfrak{g}$ is nilpotent, then $\mathcal{C}_m = \mathfrak{g}$ for some m.

Say $\mathfrak{c} = \{0\}$. Then $\{0\} = \mathcal{C}_0 = \mathcal{C}_1$.
Assume for induction that $\mathcal{C}_i = \{0\}$.

Note that quotienting by the zero ideal does nothing and we have the following: $\mathcal{C}_{i+1}$ is the ideal of $\mathfrak{g}$ such that $\mathcal{C}_{i+1} = \mathcal{C}_{i+1}/\mathcal{C}_i = \mathfrak{c}(\mathfrak{g}/\mathcal{C}_i) = \mathfrak{c}(\mathfrak{g}) = \{0\}$

Thus $\mathcal{C}_i = \{0\}$ for all i, which is a contradiction to $\mathfrak{g}$ being nilpotent.

5.1.13 Exercise 5.1.10 provides a counterexample to this.
The two dimensional Lie algebra $\mathfrak{g}$ is not nilpotent, but it has an ideal $\mathfrak{h}$ that is one dimensional, therefore nilpotent. $\mathfrak{g}/\mathfrak{h}$ is likewise one dimensional, thus nilpotent.

Chapter 6: Cartans Criteria for Solvability and Semisimplicity

6.1.2 Let Q be an inner product invariant under $Ad(G)$.

$$Q(Ad(g)X, Ad(g)Y) = Q(X, Y)$$

Let $X_1, \ldots, X_n$ be orthonormal with respect to Q.
Then $Ad(g)$is an orthogonal matrix. The Lie algebra of the Lie group $O(n)$ is $\mathfrak{so}(n)$, the space of skew symmetric matrices.
Then $ad(X)$ is skew symmetric, meaning $X_{ij} = -X_{ji}$.

$$B(X,X) = \mathrm{Tr}(ad(X) \circ ad(X)) = \sum_{i,j=1}^{n} X_{ij}X_{ji} = \sum_{i,j=1}^{n} -X_{ij}^2 = -\left(\sum_{i,j=1}^{n} X_{ij}^2\right) \leq 0$$

Furthermore, say $\mathfrak{c} = \{0\}$, $ad(X) \neq 0$ if $X \neq 0$, so for all $X \neq 0$, there is at least one $X_{ij} \neq 0$. $B(X,X) \leq -X_{ij}^2 < 0$, so B is negative definite.

Conversely, if $B(X,X) < 0$ for all $X \neq 0$, then for any such X, $ad(X)$ has at least one $X_{ij} \neq 0$, meaning $ad(X)$ is not the zero matrix, and hence $ad(X) \neq 0$, so X is not in the center. Therefore the center consists only of zero.

6.2.2 (i) Take a $\mathbb{C}$-basis of V, $\{v_1, \ldots, v_n\}$.
Note that we automatically get an $\mathbb{R}$-basis $\{v_1, \ldots, v_n, iv_1, \ldots iv_n\}$

$T(v_j) = \sum_{k=1}^{n} a_{kj}v_k$, where a_{kj} are the entries of T as an $n \times n$ $\mathbb{C}$ matrix.
If we write each a_{kj} as $a_{kj} = \alpha_{kj} + i\beta_{kj}$, then note $\mathrm{Tr}(T) = \sum_{k=1}^{n} \alpha_{kk} + i\beta_{kk}$.

$$T(v_j) = \sum_{k=1}^{n} a_{kj}v_k = T(v_j) = \sum_{k=1}^{n}(\alpha_{kj} + i\beta_{kj})v_k = \sum_{k=1}^{n} \alpha_{kj}v_k + \sum_{k=1}^{n} \beta_{kj}iv_k$$

$$T(iv_j) = \sum_{k=1}^{n} a_{kj}iv_k = T(v_j) = \sum_{k=1}^{n}(\alpha_{kj} + i\beta_{kj})iv_k = \sum_{k=1}^{n} \alpha_{kj}iv_k + \sum_{k=1}^{n} -\beta_{kj}v_k$$

Then as a real matrix, T_R is a block matrix of the form $\begin{bmatrix} [\alpha_{kj}] & [-\beta_{kj}] \\ [\beta_{kj}] & [\alpha_{kj}] \end{bmatrix}$.
Clearly then $\mathrm{Tr}(T_R) = \sum_{k=1}^{n} \alpha_{kk} + \sum_{k=1}^{n} \alpha_{kk} = 2\mathrm{Re}(\mathrm{Tr}(T))$.

(ii)
For all $x \in U^c = U \oplus JU$, we have $x = a + ib$ for $a, b \in U$.
The obvious extention of T to T_c is $T_c(x) = T(a) + iT(b)$.

Say $\{u_1, \ldots, u_n\}$ is an $\mathbb{R}$-basis of U. Then it is a $\mathbb{C}$-basis of U^c since for all x, the a and b components can be expressed as linear combinations of u_k's.

Since we can look at T with the above basis as an $\mathbb{R}$ basis and also T_c in the basis above but as a $\mathbb{C}$ basis, we are looking at the same operator on the same basis, so it will have the same trace.

6.2.3 If we take an arbitrary element X of $\mathfrak{gl}(n, \mathbb{C})$, we have the following:

$$X = \frac{X - {}^t\overline{X}}{2} + i\frac{X + {}^t\overline{X}}{2i}$$

$${}^t\overline{\left(\frac{X - {}^t\overline{X}}{2}\right)} = \frac{{}^t\overline{X} - X}{2} = -\frac{X - {}^t\overline{X}}{2} \in \mathfrak{u}(n)$$

$${}^t\overline{\left(\frac{X + {}^t\overline{X}}{2i}\right)} = \frac{{}^t\overline{X} + X}{-2i} = -\frac{X + {}^t\overline{X}}{2} \in \mathfrak{u}(n)$$

Thus we can decompose any element into a sum of a skew-hermitian matrix plus i times a skew-hermitian matrix. I.e. $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + i\mathfrak{u}(n)$.

Say $X \in \mathfrak{u}(n)$. ${}^t\overline{X} = -X$, so ${}^t\overline{iX} = (-i)(-X) = X$.
Then $i\mathfrak{u}(n)$ is made up of hermitian matrices.

Since the only matrix that is both hermitian and skew-hermitian is the zero matrix, we have $\mathfrak{u}(n) \cap i\mathfrak{u}(n) = \{0\}$. Therefore $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$, making $\mathfrak{u}(n)$ a real form of $\mathfrak{gl}(n, \mathbb{C})$.

Now, for any X, decompose it as above and apply τ, which just changes the addition in the middle into a subtraction.

$$\tau(X) = \frac{X - {}^t\overline{X}}{2} - i\left(\frac{X + {}^t\overline{X}}{2i}\right) = \frac{X - {}^t\overline{X} - X - {}^t\overline{X}}{2} = -\,{}^t\overline{X}$$

6.2.5 $\Rightarrow$

Say $B_R(x,y) = 0$ for all y.

Then $2ReB(x,y) = 0$ for all y.

$B(x,y) = \cancel{ReB(x,y)} + ImB(x,y) = Re(-iB(x,y) = \cancel{Re(B(x,iiy)} = 0.$

Since B is nondegenerate, $x = 0$, thus B_R is nondegenerate.

$\Leftarrow$

Say $B(x,y) = 0$ for all y.

Then $B_R(x,y) = 2ReB(x,y) = 0$.

Since B_R is nondegenerate, $x = 0$, thus B is nondegenerate.

6.3.4 The formula can be given by matrix equation: $Vx = C$ where x is the column vector of the $(n+1)$ values in $\mathbb{F}$, C are the coefficients of $C_0, \ldots, C_n$ writen as a column vector, and V is a Vandermonde Matrix.

The determinant of a Vandermonde Matrix is $\prod_{i<j}(x_i - x_j)$.
Since we chose the $x_0, \ldots, x_n$ to be distinct points, every $(x_i - x_j)$ is nonzero, and the determinant is nonzero, making C unique.

Hence the coefficients of our polynomial are unique, so the polynomial is unique.

6.4.8 This follows rather nicely from previous:

$$\begin{array}{rcl} \mathfrak{g} \text{ is semisimple} & \Leftrightarrow & \text{the Killing Form } B \text{ on } \mathfrak{g} \text{ is nondegenerate (Thm. 6.1.4) .} \\ & \Leftrightarrow & \text{the Killing Form } B_R \text{ on } \mathfrak{g}_R \text{ is nondegenerate (Ex. 6.2.5).} \\ & \Leftrightarrow & \mathfrak{g} \text{ is semisimple (Thm. 6.1.4).} \end{array}$$

Chapter 7: Semisimple Lie Algebras: Basic Structure and Representations

7.1.3 $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is simple.
Recall that by Cartan's critera, we only need to check that B is nondegenerate.
Let $X = X_1 + \cdots + X_n$ and $Y = Y_1 + \cdots + Y_n \in \mathfrak{g}$.

Consider $\operatorname{ad} X_i \circ \operatorname{ad} Y_j$ for $i \neq j$.
For any z this is $[X_i, [Y_j, z]]$. We have two relevant cases:
If $z \notin \mathfrak{g}_i$, then $[Y_j, z] = 0$ by the sum being direct, which makes $[X_i, [Y_j, z]] = 0$.

If $z \in \mathfrak{g}_i$, then $[Y_j, z] \in \mathfrak{g}_i$ by these being ideals, so $[X_i, [Y_j, z]]$ is the bracket of something in $\mathfrak{g}_i$ and $\mathfrak{g}_j$, so it is still zero again by the sum being direct.

Therefore, for $\operatorname{ad} X \circ \operatorname{ad} Y$, there are no mixed terms, and we can split it up amongst each of the summands.

$$\operatorname{ad} X \circ \operatorname{ad} Y = \operatorname{ad} X_1 \circ \operatorname{ad} Y_1 + \cdots + \operatorname{ad} X_n \circ \operatorname{ad} Y_n$$

Since trace is additive, when we pass to the killing form, we have that the killing form on $\mathfrak{g}$ is a sum of the killing forms on $\mathfrak{g}_i$.

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y) = \operatorname{Tr}(\operatorname{ad} X_1 \circ \operatorname{ad} Y_1 + \cdots + \operatorname{ad} X_n \circ \operatorname{ad} Y_n)$$

$$= \operatorname{Tr}(\operatorname{ad} X_1 \circ \operatorname{ad} Y_1) + \cdots + \operatorname{Tr}(\operatorname{ad} X_n \circ \operatorname{ad} Y_n) = B_{\mathfrak{g}_1}(X_1, Y_1) + \cdots + B_{\mathfrak{g}_n}(X_n, Y_n)$$

Recall that simple implies semisimple, so none of the $B_{\mathfrak{g}_i}$ are nondegenerate, thus B is nondegenerate, making $\mathfrak{g}$ semisimple.

(This can also be proven using the solvable radical by noting that any ideal of $\mathfrak{g}$ must be a direct sum of some of the $\mathfrak{g}_i$'s and bracketing).

7.2.2 First note that since $\mathfrak{g}^c$ is a direct sum, when we expand the following bracket, we have nice cancellation:

$$[x+\sigma(x), y+\sigma(y)] = [x, y] + \cancel{[x, \sigma(y)] + [\sigma(x), y]} + [\sigma(x), \sigma(y)] = [x, y] + \sigma([x, y])$$

This shows that the function $x \mapsto x + \sigma(x)$ respects the bracket operator.

It is clearly invertible, as we can decompose any y into as sum of elements from $\mathfrak{g}_1$ and $\sigma(\mathfrak{g}_1)$ and use the map $y = y' + \sigma(y'') \mapsto y'$ works as below:

$$x \mapsto x + \sigma(x) \mapsto x$$

7.3.12 Under the basis, we can write the functions as marices:

$$\pi(f) = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix}, \pi(h) = \begin{bmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{bmatrix}$$

$$\pi(e) = \begin{bmatrix} 0 & n & & \\ & 0 & 2n-1 & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

$\pi(h)\pi(f) - \pi(f)\pi(h)$ will be a matrix of entries on the subdiagonal whose entries are $(n-2j-2) - (n-2j) = -2$.
This is clearly the matrix $-2\pi(f)$.

$\pi(h)\pi(e) - \pi(e)\pi(h)$ will be a matrix of entries on the superdiagonal whose entries are $(n-2j)j(n-j+1) - j(n-j+1)(n-2j-2)$.
Factoring out j and $(n-j+1)$ gives $j(n-j+1)(\cancel{n-2j-n+2j}+2)$.
This equals $2j(n-j+1)$, which are exactly the entries of the matrix $2\pi(e)$.

$\pi(e)\pi(f) - \pi(f)\pi(e)$ will be a diagonal matrix whose entries are
$(j+1)(n-j) - j(n-j+1) = jn - j^2 + n - j - jn + j^2 - j = n - 2j$.
This is clearly the matrix $\pi(h)$.

Thus, by matrix calculations, we get the same relations as in $\mathfrak{sl}(2\ \mathbb{C})$:
$[\pi(h), \pi(f)] = -2\pi(f)$, $[\pi(h), \pi(e)] = 2\pi(e)$, and $[\pi(e), \pi(f)] = \pi(h)$,
which means that π_n is a representation.

Now, say there exists a subspace of V invariant under $\mathfrak{sl}(2\ \mathbb{C})$. Take a minimal dimensional subspace W that is invariant. It is irreducible, and we can invoke the theorem: it has a basis of eigenvectors with some relations. The choice of v_0 in the theorem is unique up to scaling, so we can take W and V to have the same first basis vector. The rest of the basis vectors were attained by successively hitting v_0 with $\pi(e)$ and stopping at the vector such that taking $\pi(e)$ once more makes it zero. This is unique, so then V and W have basis made up of the same number of power of v_0, so they have the same dimension and then must be equal.

Therefor π_n is an irreducible representation.

7.3.13 Linearity:

$$(\pi(\alpha X_1 + X_2)p)(z,w) = \left(\pi\left(\begin{bmatrix} \alpha a_1 + a_2 & \alpha b_1 + b_2 \\ \alpha c_1 + c_2 & -\alpha a_1 - a_2 \end{bmatrix}\right) p\right)(z,w)$$

$$= ((\alpha a_1 + a_2)z + (\alpha c_1 + c_2)w)\frac{\partial p}{\partial z} + ((\alpha b_1 + b_2)z - (\alpha a_1 + a_2)w)\frac{\partial p}{\partial w}$$

$$= \alpha\left((a_1 z + c_1 w)\frac{\partial p}{\partial z} + (b_1 z - a_1 w)\frac{\partial p}{\partial w}\right) + (a_2 z + c_2 w)\frac{\partial p}{\partial z} + (b_2 z - a_2 w)\frac{\partial p}{\partial w}$$

$$= \alpha(\pi(X_1)p)(z,w) + (\pi(X_2)p)(z,w)$$

Bracket Operator:

$$X_1X_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 - a_2b_1 \\ a_2c_1 - a_1c_2 & c_1b_2 + a_1a_2 \end{bmatrix}$$

$$[X_1, X_2] = \begin{bmatrix} \cancel{a_1a_2} + b_1c_2 - \cancel{a_2a_1} - b_2c_1 & a_1b_2 - a_2b_1 - a_2b_1 + a_1b_2 \\ a_2c_1 - a_1c_2 - a_1c_2 + a_2c_1 & c_1b_2 + \cancel{a_1a_2} - c_2b_1 - \cancel{a_2a_1} \end{bmatrix}$$

$$= \begin{bmatrix} b_1c_2 - b_2c_1 & 2(a_1b_2 - a_2b_1) \\ 2(a_2c_1 - a_1c_2) & c_1b_2 + -c_2b_1 \end{bmatrix}$$

$$(\pi([X_1, X_2])p)(z,w) = ((b_1c_2 - b_2c_1)z + 2(a_2c_1 - a_1c_2)w)\frac{\partial p}{\partial z} + (2(a_1b_2 - a_2b_1)z - (b_1c_2 - b_2c_1)w)\frac{\partial p}{\partial w}$$

To have this be a Lie algebra homomorphism, we need π of the bracket of X_1 and X_2 applied to a polynomial, the expression above, be equal to the bracket of the operators $\pi(X_1)$ and $\pi(X_2)$ applied to a polynomial.

First we multiply, and then we use that calculation to compute the Lie bracket:

$$\pi(X_1)\circ\pi(X_2) = \left((a_1z + c_1w)\frac{\partial}{\partial z} + (b_1z - a_1w)\frac{\partial}{\partial w}\right)\left((a_2z + c_2w)\frac{\partial}{\partial z} + (b_2z - a_2w)\frac{\partial}{\partial w}\right)$$

$$= (a_1z + c_1w)a_2\frac{\partial}{\partial z} + (a_1z + c_1w)b_2\frac{\partial}{\partial w} + (b_1z - a_1w)c_2\frac{\partial}{\partial z} + (b_1z - a_1w)(-a_2)\frac{\partial}{\partial w}$$

$$= ((a_1a_2 + b_1c_2)z + (a_2c_1 - a_1c_2)w)\frac{\partial}{\partial z} + ((a_1b_2 - a_2b_1)z + (b_2c_1 + a_1a_2)w)\frac{\partial}{\partial w}$$

Composing in the other order gives similar result with 1 and 2 switched.

$$[\pi(X_1), \pi(X_2)] = ((\cancel{a_1a_2} + b_1c_2 - \cancel{a_1a_2} - b_2c_1)z + (a_2c_1 - a_1c_2 - a_1c_2 + a_2c_1)w)\frac{\partial}{\partial z}$$

$$+((a_1b_2 - a_2b_1 - a_2b_1 + a_1b_2)z + (b_2c_1 + \cancel{a_1a_2} - b_1c_2 - \cancel{a_1a_2})w)\frac{\partial}{\partial w}$$

$$= ((b_1c_2 - b_2c_1)z + 2(a_2c_1 - a_1c_2)w)\frac{\partial}{\partial z} + (2(a_1b_2 - a_2b_1)z - (b_1c_2 - b_2c_1)w)\frac{\partial}{\partial w}$$

Apply this to any p and we get the same expression as before, and thus we have $\pi([X_1, X_2])(p)(z, w) = [\pi(X_1), \pi(X_2)](p)(z, w)$, making π a Lie algebra homomorphism.

7.3.14 Taking $v_0 = z^n$, we can use the homomorphism from problem 7.3.13. In this case, E is the matrix where $a = c = 0$ and $b = 1$.

$$\pi(E)v_0 = z\frac{\partial(z^n)}{\partial w} = 0$$

$$\pi(E)v_j = z\frac{\partial(P(n,j)z^{n-j}w^j)}{\partial w} = jP(n,j)z^{n+1-j}w^{j-1} = j\frac{n!}{(n-j)!}z^{n+1-j}w^{j-1}$$

$$= j(n-j+1)\frac{n!}{(n-j+1)!}z^{n+1-j}w^{j-1} = j(n-j+1)v_{j-1}$$

In this case, F is the matrix where $a = b = 0$ and $c = 1$.

$$\pi(F)v_0 = w\frac{\partial(z^n)}{\partial z} = nz^{n-1}w^1 = v_1$$

$$\pi(F)v_j = w\frac{\partial(P(n,j)z^{n-j}w^j)}{\partial z} = (n-j)P(n,j)z^{n-j-1}w^{j+1}$$

$$= (n-j)\frac{n!}{(n-j)!}z^{n-j-1}w^{j+1} = v_{j+1}$$

In this case, H is the matrix where $b = c = 0$ and $a = 1$.

$$\pi(H)v_0 = z\frac{\partial(z^n)}{\partial z} - w\frac{\partial(z^n)}{\partial w} = nz^n = nv_0$$

$$\pi(H)v_j = z\frac{\partial(P(n,j)z^{n-j}w^j)}{\partial z} - w\frac{\partial(P(n,j)z^{n-j}w^j)}{\partial w}$$

$$= (n-j)P(n,j)z^{n-j}w^j - jP(n,j)z^{n-j}w^j = (n-2j)v_j$$

Chapter 8: Root Space Decompositions

8.1.1 To prove this, we work by induction on k.

For $k = 0$, this is obviously true since both sides are $[x, y]$.
Assume the expansion is true for k, meaning the following holds:

$$(D - (\lambda + \mu)I_{\mathfrak{g}})^k[x, y] = \sum_{r=0}^{k} \binom{k}{r} [(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k-r} y]$$

We now work from the right side:

$$(D-(\lambda+\mu)I_{\mathfrak{g}})^{k+1}[x, y] = (D-(\lambda+\mu)I_{\mathfrak{g}}) \left(\sum_{r=0}^{k} \binom{k}{r} [(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k-r} y] \right)$$

Distribute the D using the properties of derivations to get two terms:

$$D[(D-\lambda I_{\mathfrak{g}})^r x, (D-\mu I_{\mathfrak{g}})^{k-r} y] = [D(D-\lambda I_{\mathfrak{g}})^r x, (D-\mu I_{\mathfrak{g}})^{k-r} y] + [(D-\lambda I_{\mathfrak{g}})^r x, D(D-\mu I_{\mathfrak{g}})^{k-r} y]$$

Then distribute the $-(\lambda + \mu)I_{\mathfrak{g}}$ and use linearity to get two more:

$$-(\lambda + \mu)I_{\mathfrak{g}}[(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k-r} y]$$

$$= [-\lambda I_{\mathfrak{g}}(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k-r} y] + [(D - \lambda I_{\mathfrak{g}})^r x, -\mu I_{\mathfrak{g}}(D - \mu I_{\mathfrak{g}})^{k-r} y]$$

Combine the first of each and the last of each to get two terms:
$[(D - \lambda I_{\mathfrak{g}})^{r+1} x, (D - \mu I_{\mathfrak{g}})^{k-r} y]$ and $[(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k+1-r} y]$

Now put back the binomial coefficients and carefully change the parameters:

$$\sum_{r=0}^{k} \binom{k}{r} \left([(D - \lambda I_{\mathfrak{g}})^{r+1} x, (D - \mu I_{\mathfrak{g}})^{k-r} y] + [(D - \lambda I_{\mathfrak{g}})^r x, (D - \mu I_{\mathfrak{g}})^{k+1-r} y] \right)$$

$$\sum_{r=0}^{k} \binom{k}{r} [(D-\lambda I_{\mathfrak{g}})^{r+1} x, (D-\mu I_{\mathfrak{g}})^{k-r} y] + \sum_{r=0}^{k} \binom{k}{r} [(D-\lambda I_{\mathfrak{g}})^r x, (D-\mu I_{\mathfrak{g}})^{k+1-r} y]$$

$$\sum_{r=1}^{k+1} \binom{k}{r-1} [(D-\lambda I_{\mathfrak{g}})^r x, (D-\mu I_{\mathfrak{g}})^{k+1-r} y] + \sum_{r=0}^{k} \binom{k}{r} [(D-\lambda I_{\mathfrak{g}})^r x, (D-\mu I_{\mathfrak{g}})^{k+1-r} y]$$

$$\sum_{r=0}^{k+1}\left(\binom{k}{r-1}+\binom{k}{r}\right)[(D-\lambda I_{\mathfrak{g}})^r x,(D-\mu I_{\mathfrak{g}})^{k+1-r}y]$$

Take note that $\binom{k}{-1}=\binom{k}{k+1}=0$, so we are not adding in an extra term by extending the indicies on the two sums.

Recall that binomial coefficients have a recursive formula: $\binom{k+1}{r}=\binom{k}{r-1}+\binom{k}{r}$.

$$\sum_{r=0}^{k+1}\binom{k+1}{r}[(D-\lambda I_{\mathfrak{g}})^r x,(D-\mu I_{\mathfrak{g}})^{k+1-r}y]$$

Which is exactly what we want.

8.1.4 Before we prove the statements, we have the following three remarks:

Remark $ad_{\mathfrak{gl}(V)}S$ is semisimple linear transformation since S is semisimple. We can always take a basis of V consisting of eigenvectors so that S is diagonal. $\mathfrak{gl}(V)$ has basis E_{ij}. These are eigenvectors of $ad_{\mathfrak{gl}(V)}S$ via

$$ad_{\mathfrak{gl}(V)}S(E_{ij}=[S,E_{ij}]=(\lambda_i-\lambda_j)E_{ij}$$

Remark If N is nilpotent, then so is $ad_{\mathfrak{gl}(V)}(N)$
If $N^a=0$, let $m=2a$. Then for any T, we have the following

$$\left(ad_{\mathfrak{gl}(V)}(N)\right)^m(T)=\sum(-1)^r\binom{m}{r}N^rTN^{m-r}=0$$

Remark They are polynomials!
Since ad is a homomorphism we have that S and N commute iff $ad_{\mathfrak{gl}(V)}(S)$ and $ad_{\mathfrak{gl}(V)}(N)$ commute as well.

By uniqueness of the Jordan-Chevalley decomposition, $ad_{\mathfrak{gl}(V)}(S)+ad_{\mathfrak{gl}(V)}(N)$ is the Jordan-Chevalley decomposition of $ad_{\mathfrak{gl}(V)}(X)$, and therefore $ad_{\mathfrak{gl}(V)}(S)$ and $ad_{\mathfrak{gl}(V)}(N)$ are polynomials in $ad_{\mathfrak{gl}(V)}(X)$.

(a)
Since $ad_{\mathfrak{gl}(V)}(X)$ leaves $\mathfrak{g}$ invariant, then $ad_{\mathfrak{gl}(V)}(X)|_{\mathfrak{g}} = ad_{\mathfrak{g}}(X)$.

$ad_{\mathfrak{gl}(V)}S$ is a polynomial in $ad_{\mathfrak{gl}(V)}(X)$ by our remarks, so when we restrict it to just $\mathfrak{g}$, we are taking a linear combination of powers of something that leaves $\mathfrak{g}$ invariant. The same argument holds for $ad_{\mathfrak{gl}(V)}N$

(b)
Bracketing with X_s is the semisimple part of the J.C. decomposition of $ad_{\mathfrak{g}}X$. By uniqueness of the J.C. decomposition, we must have $ad_{\mathfrak{gl}(V)}(S)|_{\mathfrak{g}} = ad_{\mathfrak{g}}(X_s)$

for any $Y \in \mathfrak{g}$ $[S, Y] = ad_{\mathfrak{gl}(V)}S(Y) = ad_{\mathfrak{g}}X_s(Y) = [X_s, Y]$, by the above, and therefore $[S - X_s, Y] = 0$ by linearity.
So $S - X_s$ commutes with everything in $\mathfrak{g}$, and therefore is in the centralizer.

The same argument works for $N - X_n$.

(c)
S is a polynomial in X, so $[S, X_s] = [\sum a_j X^j, X_s] = \sum a_j [X^j, X_s]$.
$[X^j, X_s] = -ad_{\mathfrak{g}}X_s(X^j)$, which must be zero, since X_s is semisimple and commutes with X.

The same argument works for N and X_n.

(d)
Since S and N are polynomials in X and this holds for X, it automatically holds for S and N.

(e)
$[S - X_s, \mathfrak{g}] = 0$, so for all $T \in \mathfrak{g}$, we have that $(S - X_s) \circ T = T \circ (S - X_s)$. This means $S - X_s$ intertwines the $\mathfrak{g}$ action on V_i, so by Schur's Lemma it is a scalar operator. The same argument works for $N - X_n$.

(f)
X_s can be broken down as $X_s = (X_s - S) + S$. By (d), S leaves V_i invariant and is semisimple, whereas $(X_s - S)$ is a scalar operator by (e), and is then semisimple. The sum of two semisimples is semisimple, so X_s is semisimple.

(g)
Recall that $\mathfrak{g}$ is semisimple, so $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then any Y is a linear combination of Lie brackets, which have zero trace. Trace respects linear combinations, therefore $\text{Tr}(Y) = 0$.

(h)
As we proved in (e), $N - X_n$ is a scalar operator, i.e. $N - X_n = \lambda I$.
A nilpotent linear transformation has only zero as an eigenvalue, so $\mathrm{Tr}(N) = 0$.
X_n is in $\mathfrak{g}$, so by (g), $\mathrm{Tr}(X_n) = 0$.
$\lambda^n = \mathrm{Tr}(N - X_n) = \mathrm{Tr}(N) - \mathrm{Tr}(X_n) = 0$, therefore $\lambda = 0$, thus $N - X_n = 0$.

Again, we proved in (e), $S - X_s$ is a scalar operator, i.e. $N - X_n = \mu I$.
Now, $X = S + N = X_s + X_n$. Rearranging gives $S = X_s + X_n - N = X - N$.
By part (g), $\mathrm{Tr}(X) = 0$ and by nilpotency, $\mathrm{Tr}(N) = 0$, hence $\mathrm{Tr}(S) = 0$.
We also know, by (g), that $\mathrm{Tr}(X_s) = 0$.
Combining this gives us $\mu^n = \mathrm{Tr}(S - X_s) = \mathrm{Tr}(S) - \mathrm{Tr}(X_s) = 0$, therefore $\mu = 0$, thus $N - X_n = 0$.

So we have $N = X_n$ and $S = X_s$, making the Jordan-Chevalley and abstract Jordan-Chevalley decomposition equal.

8.3.5 To do soon